$$\dot{r} = p/m$$
$$T(t, q, p) = \tfrac{1}{2}m(p^2/m^2 + \omega^2 r^2 \sin^2\theta)$$
$$b(t, q, p) = \tfrac{1}{2}(p^2/m) - \tfrac{1}{2}m\omega^2 r^2 \sin^2\theta$$

Hamilton's equations are

$$\dot{r} = p/m \tag{36}$$

$$\dot{p} = \frac{\partial h}{-\partial r} + (\dot{m}u_j + F_j)\frac{\partial x_j}{\partial r}$$
$$= (m\omega^2 \sin^2\theta)r - \dot{m}c + \dot{m}\dot{r} - mg\cos\theta \tag{37}$$

Eliminating $\dot{p}$ in (37) by means of (36), one obtains the differential equation

$$m\ddot{r} - (m\omega^2 \sin^2\theta)r = -\dot{m}c - mg\cos\theta$$

which was obtained previously by the Lagrangian method.

Interaction of Magnetohydrodynamic Simple Waves in Monatomic Fluids

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It is well known that the differential equations governing the one-dimensional nonsteady flow of an ideal, inviscid, compressible fluid may be reduced to a single second-order, linear, partial differential equation, the Euler-Poisson equation, whose general solution may be given explicitly in terms of two arbitrary functions for a set of values of the adiabatic index which includes those of usual physical interest.[1]

It is the purpose of this note to show that the interaction of hydromagnetic simple waves may be reduced to a single second-order, linear, partial differential equation, and the Riemann function thereof may be found from a Fourier superposition of the solutions of a linear, second-order, ordinary differential equation. Although it does not seem possible to solve this latter in terms of known functions, the advantage of the present method over other numerical methods, e.g., finite differences, is that the Riemann function is to be determined once and for all, and, thereafter, the solutions to specific initial-value problems may be given by quadratures.

The isentropic one-dimensional nonsteady motion of an ideal, inviscid, perfectly conducting monatomic compressible fluid subjected to a transverse magnetic field, i.e., the induction $\mathbf{B} = (0,0,B)$, is governed by the system of equations:

$$c_t + uc_x + (cu_x/3) = 0 \tag{1}$$

$$3cc_x + u_t + uu_x + b^2(B_x/B) = 0 \tag{2}$$

$$B_t + uB_x + Bu_x = 0 \tag{3}$$

where u, c, ρ, $b^2 = B^2/\mu\rho$, and μ are, respectively, the particle velocity, local speed of sound, density, square of the Alfvén speed, and permeability. Partial derivatives are denoted by subscripts, and all dependent variables are functions of x and t alone.

From the characteristic form of Eqs. (1–3), it has been shown[2, 3] that B/c^3 is constant along each particle path, and for a constant state or a simple-wave flow, B/c^3 was constant throughout the flow. This result still will obtain in the region of interaction of two simple waves, since the particle paths therein originate either in a constant state or a simple wave. Then if $B = r_1c^3$, $\rho = r_2c^3$, with constants r_1 and r_2, $b^2 = r_1{}^2c^3/\mu r_2 \equiv kc^3$, so that $\omega^2 = b^2 + c^2 = c^2(1 + kc)$, and the system of Eqs. (1–3) may be replaced by the system

$$x_\beta = (u + \omega)t_\beta \tag{4}$$

$$x_\alpha = (u - \omega)t_\alpha \tag{5}$$

$$u/2 + (1 + kc)^{3/2}/k = u/2 + (\omega/c)^3/k = \alpha \tag{6}$$

$$-u/2 + (1 + kc)^{3/2}/k = -u/2 + (\omega/c)^3/k = \beta \tag{7}$$

where (α, β) may be considered as generalizations of the usual Riemann invariants. In terms of (α,β), it was shown that

$$u = \alpha - \beta \tag{8}$$

$$kc = [k(\alpha + \beta)/2]^{2/3} - 1 \tag{9}$$

$$\omega = (\alpha + \beta)/2 - [k(\alpha + \beta)/2]^{1/3}/k \tag{10}$$

and from Eqs. (8–10) that an explicit solution for a centered simple wave could be given.[4]

The second-order, linear, partial differential equation is obtained by eliminating x from Eqs. (4) and (5) and using the relations given by Eqs. (8) and (10). This gives

$$t_{\alpha\beta} + \frac{(9\tau^{2/3} - 1)(t_\alpha + t_\beta)}{6(\alpha + \beta)(\tau^{2/3} - 1)} = 0 \tag{11}$$

where $\tau = k(\alpha + \beta)/2$. It is convenient to introduce the new dependent variable w by requiring that[5] $w_\alpha = x - (u + \omega)t$ and $-w_\beta = x - (u - \omega)t$, so that $w(\alpha, \beta)$ satisfies the equation

$$w_{\alpha\beta} + \frac{(3\tau^{2/3} + 1)(w_\alpha + w_\beta)}{6(\alpha + \beta)(\tau^{2/3} - 1)} = 0 \tag{12}$$

In order to transform Eq. (12) into a form for which results are known, a new dependent variable $v(\alpha, \beta)$ is introduced through the substitution $w = (\alpha + \beta)^{1/6}v(\alpha, \beta)/(\tau^{2/3} - 1)$. The resultant equation for v is

$$v_{\alpha\beta} + \left[\frac{9\tau^{2/3} + 7}{36(\alpha + \beta)^2(\tau^{2/3} - 1)}\right]v = 0 \tag{13}$$

The Riemann function of Eq. (13) may be obtained from the following result, which was derived heuristically by Riemann[5] and proved rigorously by Cohn.[6] Given the equation

$$v_{\alpha\beta} + H(\alpha + \beta)v = 0 \tag{14}$$

let $\xi = \alpha + \beta$, $\eta = \alpha - \beta$, and

$$R^* = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp[i\nu(\eta - \eta')]y(\xi, \xi'; \nu)d\nu$$

where $y(\xi,\xi'; \nu)$ is defined by the ordinary differential equation

$$(d^2y/d\xi^2) + [\nu^2 + H(\xi)]y = 0 \tag{15}$$

with $y = 0$, $dy/d\xi = 1$ for $\xi = \xi'$. Then, changing back to the variables (α, β), R^* satisfies the relation

$$R^* = \tfrac{1}{2}[\mathrm{sgn}(\alpha - \alpha') + \mathrm{sgn}(\beta - \beta')]R(\alpha, \beta;\ \alpha', \beta')$$

where $R(\alpha, \beta;\ \alpha', \beta')$ is the Riemann function for Eq. (14).

From this result, the Riemann function of Eq. (13) may be obtained from the Fourier superposition of the solutions of

$$\frac{d^2y}{d\xi^2} + \left[\nu^2 + \frac{9\sigma\xi^{2/3} + 7}{36\xi^2(\sigma\xi^{2/3} - 1)}\right]y = 0 \tag{16}$$

where $\sigma = (k/2)^{2/3}$, subject to the subsidiary conditions appended to Eq. (15). In order to put Eq. (16) into a form more convenient for comparison with known equations, let $\zeta = \sigma\xi^{2/3}$ and $y(\xi) = \zeta^{-1/4}s(\zeta)$. Then

$$\zeta(\zeta - 1)s'' + (1 - \zeta)s' + s[1 + \tfrac{9}{4}(\nu^2/\sigma^3)\zeta^2(\zeta - 1)] = 0 \tag{17}$$

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the normal form of which is obtained by the substitution $s(\zeta) = \zeta^{1/2}q(\zeta)$ with the result

$$\frac{d^2q}{d\zeta^2} + \left[-\frac{3}{4\zeta^2} + \frac{1}{\zeta(\zeta-1)} + \frac{9}{4}\frac{\nu^2\zeta}{\sigma^3} \right] q = 0 \quad (18)$$

It has not been possible to relate Eqs. (17) and (18) to any known equations, but the following observations are somewhat suggestive. For $\nu = 0$, Eq. (17) reduces to the hypergeometric equation, and for very large ζ, Eq. (18) reduces to one that has Airy functions as solutions. Although a large number of transformations has been tried in an attempt to exploit these observations, e.g., solutions as products of known functions, no usable results have been obtained.

In summary, it is felt that the indicated Fourier superposition of the solutions of Eq. (16) provides a numerical algorithm for determining the requisite Riemann function, but Eq. (17) remains a worthy candidate for further research.

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Beam Current Measuring Device for Ion Engine Research

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THE ion beam exiting from an ion rocket usually is measured by metering the electron ground current. Beam current measured in this way usually is correct, but no information concerning the partially neutralized or remaining net current is obtained.

An instrument to measure net direct-current at any position along an ion beam is the magnetic ammeter that detects the magnetic field created by the flow of charged particles. The feasibility of this concept was demonstrated in Ref. 1, where the current of an electron beam passing through a 6-cm-diam mumetal ring was measured. The present study was undertaken to find out if a much larger instrument would have sufficient sensitivity to prove useful in ion engine research.

As shown in Fig. 1, the ammeter consists of an indium arsenide probe mounted in the air gap of a high permeability ring. The voltage signal from the Hall effect transducer is detected with a commercial gaussmeter.

The current of ions passing through the ring can be considered to be a one-turn coil, and the magnetic field in the air gap is calculated from the magnetic circuit equation. For a small gap, the magnetic flux Φ is constant around the ring, and[4]

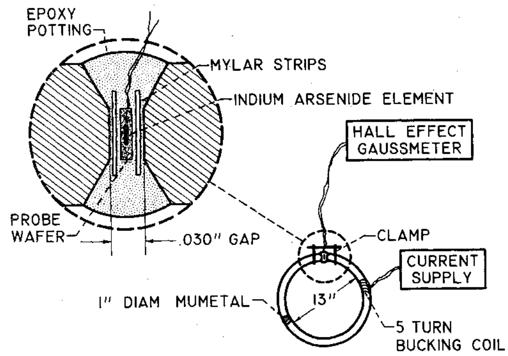

Fig. 1 Mumetal ring assembly

$$\Phi \oint \frac{dl}{\mu A} = I \quad (1)$$

which can be written as

$$\Phi[(l_0/\mu_0 A_0) + (l_m/\mu_m A_m)] = I \quad (2)$$

where l_0, A_0, and μ_0 are the length, area, and permeability of the air gap, and l_m, A_m, and μ_m refer to the same quantities in the metal ring.

The sensitivity of the device can be defined as the ratio of the magnetic field in the air gap to the current passing through the ring:

$$S = B_0/I \quad (3)$$

where $B_0 = \Phi/A_0$. From (2)

$$S = \frac{1}{(l_0/\mu_0) + (l_m A_0/\mu_m A_m)} \quad (4)$$

In the present case $(l_0/\mu_0) >> (l_m A_0/\mu_m A_m)$, so that the sensitivity depends only on the characteristics of the air gap. The calculated sensitivity of the present device $S = \mu_0/l_0 = 0.02$ gauss/ma is in agreement with the measured value. The ring can be made indefinitely large and still have the sensitivity dependent only on air gap characteristics as long as

$$(l_m/l_0)(\mu_0/\mu_m)(A_0/A_m) << 1 \quad (5)$$

For an area ratio A_0/A_m of unit order, the length ratio l_m/l_0 should be much less than the permeability ratio μ_0/μ_m. If the inequality (5) is not satisfied, the device will require a more careful calibration.

In practice the magnetic field is not measured directly. Current I_c through the N turn bucking coil is used to cancel the field in the ring. At the null condition the beam current is NI_c or $5I_c$ for the five-turn bucking coil used.

The greatest inaccuracy in the device is d.c. drift caused by the extreme sensitivity of the probe to temperature. When a

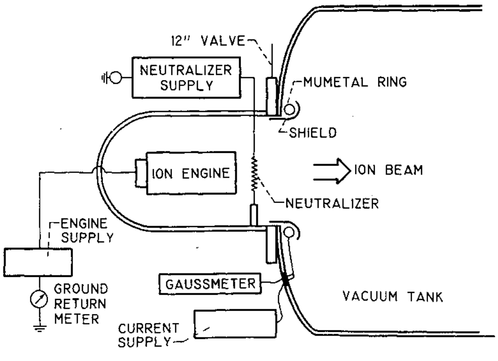

Fig. 2 Test setup

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